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The planar Turán number of double star $S_{2,4}$. (English. English summary)

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All graphs considered are simple and undirected. The planar Turán number $\text{ex}_{\mathcal{P}}(n, H)$ of a graph H is the maximum number of edges in an H -free planar graph on n vertices. A (k, l) -star, denoted $S_{k,l}$, is the graph obtained from an edge by joining its end vertices to k and l vertices, respectively.

By the work of D. Ghosh et al. [“Planar Turán number of double stars”, preprint, arXiv:2110.10515], it is known that $\text{ex}_{\mathcal{P}}(n, S_{2,2}) = 2n - 4$ for $n \geq 16$, $\text{ex}_{\mathcal{P}}(n, S_{2,3}) = 2n$ for $n \geq 1$, and $15n/7 \leq \text{ex}_{\mathcal{P}}(n, S_{2,4}) \leq 8n/3$ for $n \geq 1$. In this article, the authors show that if G is an $S_{2,4}$ -free planar graph on n vertices, then $e(G) \leq 31n/14$, with equality when $n \equiv 0 \pmod{14}$.

The proof works by analyzing the structure of $S_{2,4}$ -free planar graphs G on n vertices that have minimum degree $\delta(G) \geq 3$ and maximum degree $\Delta(G) \leq 5$. A small number of cases are considered based on the location of the vertices of maximum degree to show that any such graph satisfies $e(G) \leq \lfloor 31n/14 \rfloor$. Moreover, two graphs on 7 vertices and one on 8 vertices are constructed such that the upper bound is attained. The authors describe a procedure of piecing together these graphs in a tree-like fashion to produce infinite families of $S_{2,4}$ -free planar graphs that attain the upper bound.

Some other recent works on the planar Turán number of $S_{k,l}$ -free graphs include improved upper bounds for $S_{2,5}$ -free graphs by X. Xu, Y. Hu and X. Zhang [Discrete Appl. Math. **358** (2024), 326–332; MR4778796], the exact value for $S_{3,3}$ -free graphs by Xu et al. [“Planar Turán number for balanced double stars”, preprint, arXiv:2406.05758], and the exact value for $S_{3,4}$ -free graphs by Xu, Zhang and J. Shao [AIMS Math. **10** (2025), no. 1, 1628–1644; MR4854366].

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.